Homoclinic Dynamics: A Scenario for Atmospheric Ultralow-Frequency Variability

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ABSTRACT

In this paper, a link will be established between atmospheric ultralow-frequency variability (ULFV) and the occurrence of homoclinic dynamics in models of large-scale atmospheric flow. It is known that uncoupled atmosphere models possess significant variability on very long timescales (years to decades), which must be generated by internal atmospheric dynamics. The mathematical structure of this long-timescale variability is investigated, using a global two-layer atmosphere model formulated in terms of preferred flow patterns (EOFs). Due to its efficient formulation, this model can simulate an atmospheric flow with realistic features, using only a small number of degrees of freedom. The 10-dimensional version of the model possesses both nonzero ultralow-frequency variability and several realistic short timescales. The essence of the ultralong timescale behavior of the 10D model, which manifests itself as bursting in the atmospheric turbulent energy, can be represented by a four-dimensional subsystem. In this subsystem, strong evidence for the existence of a homoclinic orbit is found. The chaotic dynamics generated by the homoclinic orbit explains the occurrence of ultralong timescales in the model. It is argued that hints of homoclinic dynamics can also be found in more complex models. As an example, a T21 barotropic atmosphere model (231 dimensions) of the Northern Hemisphere is shown to possess behavior that suggests the existence of a homoclinic orbit.

1. Introduction

Atmospheric variability on very long timescales, purely generated by its own internal dynamics, is not a very well understood feature of the atmosphere. The short timescales (up to a week) are the classical subject of meteorological research, since they are within the typical reach of weather prediction. Changes in the atmosphere on timescales from a week to a few months, often identified as “low-frequency variability” or LFV, have been studied rather intensely in recent decades using and studying concepts such as blocking, multiple equilibria, and teleconnection patterns. In contrast, atmospheric behavior on timescales beyond several months (also called ultralow-frequency variability or ULFV) has hardly received any attention apart from the context of coupled models: atmospheric ULFV is usually associated with other, “slower” components of the climate system (e.g., the oceans) and is therefore mostly studied in models in which the atmosphere is coupled to the oceans, to sea ice, etc. The ULFV of the atmosphere in itself, and its internal dynamics has been ignored for a long time.

The limited number of papers that have been devoted to the ULFV generated by the internal dynamics of realistic atmosphere models mostly have a descriptive/phenomenological content: using models of varying complexity, power spectra of the frequency domain are calculated, showing nonzero variability on very long timescales (years and longer). James and James (1989) use a 5-level, T21 primitive equation model with both constant and periodic forcing to arrive at power spectra most of which show peaks at decadal periods over a red-noise type background. James and James (1992), using the same model, find maximum variability in a broad frequency-band, roughly between 5 and 40 yr. They also investigate the spatial structure of the ULFV in their model, using EOFs of the zonal-mean zonal flow. From this information, they hypothesize that the interactions between zonal flow and baroclinic waves induce the long timescale behavior. This hypothesis is further investigated (and supported) in James et al. (1994). Pielke and Zeng (1994) run the Lorenz 1984 model for 1100 yr and find nonzero spectral power for periods longer than a year. Kurgansky et al. (1996) use a low-order (eight components) hemispheric, baroclinic model and find maximum variability between 3 and 44 yr from a 1100-yr integration. Dethloff et al. (1998) run a 20-dimensional baroclinic model with orography for 10,000 yr, thus obtaining results with better statistical reliability than in previous studies. They find significant variability at ultralow frequencies, with a maximum at a period of about 10 yr. Handorf et al. (1999), as part of their study of a coupled atmosphere–ocean model, run a moderately complex at-
mosphere model with prescribed, seasonally varying sea surface temperatures for 1000 yr and find modes that are strongly peaked at 9 yr.

The above studies follow roughly the same strategy: a relatively simple atmospheric model, which allows for computationally cheap and fast integrations of considerable length, is integrated for a very long time (up to 10 000 yr). The data thus obtained are used to calculate power spectra of various quantities, such as spectral components. They all show significant spectral power on very long timescales, orders of magnitudes longer than the typical response timescale of the atmosphere.

In this paper, another approach is followed. Rather than aiming at more detailed information about the precise spatiotemporal structure of the atmospheres ultralow-frequency variability, we want to focus on the mathematical structure and mechanism(s) underlying the ultralong timescale behavior of atmospheric models. Using techniques and concepts from dynamical systems theory we shall try to gain more insight into the generation of ULFV in the atmosphere. This goal puts us in the middle of the gap between mathematics—dynamical systems on the one hand and meteorology—climatology on the other hand. For mathematicians to be able to analyze them, nonlinear systems must be low-dimensional; however, in order to be somewhat realistic, atmosphere models cannot be that low-dimensional. The famous Lorenz models of 1963 and 1984, for instance, are often thought of as metaphors of the real atmosphere. However, even though these models are inspired by meteorological problems, they are not even remotely capable (nor meant) to describe global atmospheric dynamics. The (many) results that are known by now on the creation of chaos and strange attractors in these models are not necessarily valid for global atmospheric models, and the insights the Lorenz models provide should therefore only be applied with great care to the global atmosphere. The mechanisms and sources of chaos and long timescales can be totally different in global models.

In order to combine a small number of variables with an acceptable degree of realism, we have turned toward the efficient model of Achatz and Branstator (1999): an atmospheric model formulated in terms of empirical orthogonal functions (EOFs), closed with an empirical closure scheme. From a GCM dataset [National Center for Atmospheric Research Community Climate Model (NCAR CCM) version 0B] EOFs were calculated and used as basis functions for a two-layer model based on filtered [quasigeostrophic (QG)-like] equations. After truncation of this model, an empirical closure scheme was applied in which forcing and linear terms of the model equations were changed in order to minimize tendency errors produced by the model. With as few as 30 degrees of freedom, the resulting model has a remarkably good performance when it comes to climate simulation. We shall work with the 10-dimensional (10D) version of this model, as it is a low-dimensional version that still shows both nonzero variability on ultralong timescales and several realistic short timescales. The 10D model will be analyzed in order to find out what mechanism(s) and source(s) create chaos and long timescales (in other words: ultralow-frequency variability). Due to the degree of realism of the 10D model, and its derivation from a GCM, the results obtained on the creation of chaos and ULFV are more likely to represent realistic atmospheric characteristics than the results on the Lorenz models (or any other model analysis we know of). To our knowledge, no attempt has been made before to gain more insight into the creation of chaos in an atmosphere model that is designed to be global and realistic, rather than to be merely a metaphor.

This paper is organized as follows. In section 2, the derivation and formulation of the EOF model is briefly described. In section 3 we compare the 10-EOF model with the 30-EOF model and the GCM, in order to get an impression of its degree of realism. Section 4 contains an analysis of the 10-EOF model. We will show that the essence of the long timescale behavior can be represented by a four-dimensional subsystem of the 10D system; this is discussed in section 5, which also contains results of the analysis of the 4D subsystem. In section 6 we will discuss to what extent the obtained results and insights can be expected to be valid for more complex models, such as the 30-EOF model and the GCM. In section 7, the output of an entirely different model, a T21 standard barotropic model, will be used to further support the relevance of the results for more complex models. Section 8 concludes the paper with a summary and outlook.

2. Model formulation

The atmospheric model we wish to analyze has to meet two conflicting requirements: the model must be low-dimensional, and it must be realistic. Although it is impossible to resolve this conflict entirely, one can at least choose to work with a model that uses its few degrees of freedom as efficiently as possible, thereby providing an acceptable compromise. Such efficiency can be created by describing atmospheric flow in terms of preferred flow patterns rather than in terms of spherical harmonics. The model formulated by Achatz and Branstator (1999, AB hereafter) employs empirical orthogonal functions (EOFs) as its basis functions. Such a basis captures a maximum amount of atmospheric variability with a limited number of basis functions. In combination with an empirical closure scheme, a highly efficient model is obtained. This model will be the subject of our investigation; its derivation will be described briefly in this section. For more detailed information on the derivation the reader is referred to AB.

The recipe used by AB in order to make their model efficient consists of three steps. First, streamfunction EOFs are calculated from GCM data. Then the expan-
sion coefficients of these EOFs are used as the variables of a two-layer filtered (QG-like) model. After truncating this EOF-based model, an empirical closure scheme is applied in which the linear and constant terms of the filtered equations are modified such that the resulting set of equations minimizes the tendency error for states in the GCM dataset.

The GCM data used are two 50 000-day integrations (half-daily output), under perpetual January conditions, performed with version 0B of the NCAR Community Climate Model (an R15 model with nine levels in the vertical). Time series describing the streamfunction at 200 and 700 mb are extracted from the data; the atmospheric state at time \( t \) is now represented by a 990-dimensional vector \( \mathbf{X}(t) \), which contains the spectral coefficients of the streamfunction at both 200 and 700 mb. The EOFs are calculated from the time series \( \{ \mathbf{X}(t) \} \) using a total energy norm, yielding a set of patterns \( \{ \mathbf{p}^i \}, i = 1, \ldots, 990 \). Each EOF corresponds to a streamfunction anomaly pattern at both 200 and 700 mb. The EOF spectrum is rather flat: about 500 EOFs are needed to explain 90\% of the total energy variance.

The two-layer filtered model used by AB was derived by Lorenz (1960); it is essentially the same as a classical quasigeostrophic model. Projection of the (discretized) model equations onto the EOFs, that is, replacing the state vector \( \mathbf{X}(t) \) by \( \bar{\mathbf{X}} + \Sigma \mathbf{a}(t) \mathbf{p}^i \), yields an equation for \( \mathbf{a} \), the 990-dimensional vector containing the EOF expansion coefficients \( a_i \):

\[
\dot{\mathbf{a}} = \mathbf{F}^a + \mathbf{L}^a \mathbf{a} + \mathbf{N}(\mathbf{a}, \mathbf{a}) \tag{1}
\]

with time-independent forcing \( \mathbf{F}^a \), linear operator \( \mathbf{L}^a \), and quadratic nonlinearities \( \mathbf{N}(\mathbf{a}, \mathbf{a}) \). This system is truncated, leaving an \( m \)-dimensional set of ordinary differential equations (ODEs) prescribing the time evolution of \( \mathbf{a} = (a_1, \ldots, a_m)^T \).

Finally, \( \mathbf{F}^a \) and \( \mathbf{L}^a \) are replaced by new operators \( \mathbf{F} \) and \( \mathbf{L} \), which were determined such that the resulting set of equations produces tendencies \( \dot{\mathbf{a}} \) which resemble the corresponding tendencies in the GCM data as close as possible. This empirical correction of \( \mathbf{F} \) and \( \mathbf{L} \) accounts for the effect of unresolved physics and dynamics on the explicitly described patterns, for example, the effect of the EOFs that were lost in the truncation, or dynamical processes in the Tropics that are not captured by the QG equations (thereby making the model more realistic than ordinary QG models). The final result is an \( m \)-dimensional set of ODEs:

\[
\dot{\mathbf{a}} = \mathbf{F} + \mathbf{L} \mathbf{a} + \mathbf{N}(\mathbf{a}, \mathbf{a}) \tag{2}
\]

Note that (2) models the time evolution of deviations of the atmosphere from the GCM mean state \( \bar{\mathbf{X}} \). Anomalous, not total flow is described; the “background” flow \( \bar{\mathbf{X}} \) is given and constant. The atmospheric mean state resulting from the model (2) can be different from the mean state in the GCM, so \( \Sigma_{mn} \mathbf{p}^i \mathbf{p}^j \) can be nonzero.

A last remark on the energy in the model: the total turbulent energy (the phase space distance to \( \mathbf{a} = 0 \) using the total energy norm) is equal to the sum of squares of the EOF coefficients, that is, \( E = \mathbf{a} \cdot \mathbf{a} \) (Euclidean inner product). This is due to the use of the total energy norm in the calculation of the EOFs. The energy \( E \) is unaffected by the nonlinear terms, which means that \( \mathbf{a} \cdot \mathbf{N}(\mathbf{a}, \mathbf{a}) = 0 \). The nonlinear interactions redistribute energy but do not change the overall energy in the system (this is a common property of fluid dynamical models—the advection terms conserve energy).

3. The 10D model

The performance of the empirical EOF model was tested in various ways by AB, for example, by inspecting the climatology of the 500-EOF model, by comparison with an optimized standard spectral model and by calculation of tendency errors for EOF model versions of varying dimension (using the GCM data as reference). The last test showed that the relative tendency error was quite large (0.9–0.4) but monotonically decreasing for increasing number of dimensions. In contrast, the relative first moment error and relative second moment error showed minima \([O(10^{-2})]\) in the range between roughly 30 and 70 dimensions. For higher dimensions, these error measures increased. Apparently, quite simple EOF models (e.g., 30-dimensional), although not describing the instantaneous tendency very well, can capture a lot of the dynamics on longer timescales (reflected in the low first and second moment errors). This feature makes these models suitable for studying (ultra-) low-frequency variability of the atmosphere.

The analysis in this paper will focus on the 10-dimensional EOF model. The relative tendency error and the relative first moment error of the 10D model are comparable to those of the 30D model; its relative second moment error is substantially larger. The total energy variance explained by the first 10 EOFs is 21.7\%; 30 EOFs explain 34.9\%. Integrations of the 10D and 30D models and inspection of the time series of the turbulent energy resulting from these integrations shows (Fig. 1) that both models exhibit large variations in energy: periods of relative quietness (low energy) and sudden outbursts of energy during which the atmosphere is highly active. The same phenomenon can be seen in the GCM data if we restrict ourselves to the large spatial scales, that is, to the leading EOFs. In Fig. 1 the turbulent energy of the GCM data projected onto the first 30 EOFs is shown. The bursting-like behavior is the dominant aspect of the long timescale behavior of all three models. The energy bursts become farther apart in time and more pronounced, compared to the GCM, in the 30D model and especially in the 10D model. One could say that the strong, characteristic bursts in the 10D model are schematized, overpronounced versions of the bursts in the 30D model and the GCM. Their heights, however, are of the same magnitude.

Power spectra of the turbulent energy of the 30D and
10D models, calculated from integrations of $3 \times 10^6$ days length ($8 \times 10^3$ yr), are provided in Fig. 2. No turbulent energy spectrum of the GCM data is shown, due to the limited time span of that data. The spectrum of the 10D model clearly shows nonzero variability at very low frequencies. It also shows a strong peak at a frequency of about 0.15 cycles per year (i.e., a period of about 6.7 yr); this peak is due to the bursting behavior of the model. The 30D model spectrum has the same characteristics: nonzero ULFV and a maximum at periods slightly less than 10 yr (although this maximum is much less pronounced than in the 10D model). Apparently, the strong 6.7-yr peak in the 10D model is much too pronounced but not unrealistic, since its frequency matches both with the 30D model and with peaks in the frequency spectra of previous studies: Handofer et al. (1999) find a peak of 9 yr, Dethloff et al. (1998) find a decadal peak, and both James and James (1989, 1992) and Kurgansky et al. (1996) find maximal variability at decadal periods.

The above observations suggest the validity of using the 10D EOF model for studying the mechanisms of atmospheric ULFV. It is simple and yet describes global atmospheric dynamics; it has some degree of realism due to its derivation from a GCM. The model possesses “short” timescales that are also prominent in the 30D model and the GCM; on long timescales it shows nonzero variability associated to bursting behavior also present in both the 30D model and the GCM. Furthermore, its dimension is low enough to allow for the use of...
4. Analysis of the 10D model

In the previous section, it was already noted that the 10D model behavior is marked by outbursts of turbulent energy; phases with rather regular behavior and low turbulent energy (so-called “laminar” phases) are followed by phases with chaotic behavior and high turbulent energy, as can be seen in Fig. 1. The regularity of the behavior at low energies indicates the presence of an unstable equilibrium solution of the model equations (2) close to $a = 0$.

The equilibrium (or fixed point) can be found by continuation (starting at $a = 0, F = 0$), using, for example, the continuation software-package AUTO (Doedel and Wang 1995), or using a rootfinder. With both methods the equilibrium point $a_{eq}$ is easily found. Linearization shows that it is a saddle-type fixed point, with 1 unstable and 4 stable complex pairs of eigenvalues. In Table 1 the eigenvalues are listed, together with their associated periods $2\pi/|1\lambda_i|$ and $e$-folding times $1/|Re\lambda_i|$. The periods of the five eigenmodes explain most of the peaks in the EOF1 power spectrum (Fig. 3): those at frequencies of 2.0, 5.8, 11.5, and 18.6 cycles per year match very well with the periods of modes 5, 4, 3, and 2, respectively. The period of mode 1 is not visible in the EOF1 power spectrum but can be observed in the power spectra of most other EOFs (not shown).

The atmospheric patterns associated to the eigenmodes are shown in Fig. 4. Mode 1 is a Northern Hemispheric wavenumber-5 zonal wave at midlatitudes. Modes 2 and 3 are equatorial waves; they are manifestations of the models’ representations of the Madden–Julian oscillation (MJO), also known as the 30–60-day tropical oscillation. Mode 2 has wavenumber 2 [an aspect of the MJO that has been observed before (Lau and Peng 1987; Salby and Hendon 1994) but that is less well known than the wavenumber-1 pattern of the MJO]; it also shows some North Pacific activity. Mode 3 is a wavenumber-1 equatorial wave, the most famous aspect of the MJO (see Von Storch et al. 1988). Modes 4 and 5 are both largely Northern Hemispheric; they resemble the Pacific–North American (PNA) pattern. Mode 5 also shows an alternation between blocked and zonal flow in the North Atlantic region.

The $e$-folding times of the five eigenmodes may seem somewhat strange. They are one to two orders of magnitude larger than the most commonly known timescales associated to the growth and decay of atmospheric perturbations (typically a few days to a few weeks). This is due to the empirical nature of the model under consideration. Growth and decay cannot be attributed to physical processes (such as baroclinic instability or Ekman dissipation) anymore; rather, they combine the cumulative effect of many such processes. The empirical closure scheme applied by AB results in a model that describes the net effect of many different physical processes on the resolved patterns.

Another peculiarity of the model stems from the fact that it describes deviations from the mean flow instead of total atmospheric flow. By shifting the variables, say $\psi$, to $\psi_{c} = \psi - \overline{\psi}$ (with $\overline{\psi}$ the mean state), the quadratic nonlinearities (advection terms) $N(\psi, \psi)$ split into a nonlinear part $N(\psi_{c}, \psi_{c})$, a constant part $N(\overline{\psi}, \overline{\psi})$ and a linear part $2N(\overline{\psi}, \psi_{c})$. The latter is added to the “old” linear operator, which contained, for example, Coriolis terms, friction, and dissipation. The new linear operator can have growing eigenmodes, whereas the old operator was purely dissipative. The energy transfer between the mean flow and its disturbances, which is modeled with nonlinear terms in total flow

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**Table 1. Eigenvalues of linearized 10D EOF model.**

<table>
<thead>
<tr>
<th>No.</th>
<th>Eigenvalue</th>
<th>Period</th>
<th>$e$-folding time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.002721 \pm 0.438839 i$</td>
<td>14.3 days</td>
<td>367.4 days</td>
</tr>
<tr>
<td>2</td>
<td>$-0.002230 \pm 0.319469 i$</td>
<td>19.7</td>
<td>448.4</td>
</tr>
<tr>
<td>3</td>
<td>$-0.001684 \pm 0.198707 i$</td>
<td>31.6</td>
<td>593.8</td>
</tr>
<tr>
<td>4</td>
<td>$-0.002863 \pm 0.097667 i$</td>
<td>64.3</td>
<td>349.2</td>
</tr>
<tr>
<td>5</td>
<td>$0.001655 \pm 0.0353438 i$</td>
<td>177.8</td>
<td>604.1</td>
</tr>
</tbody>
</table>
models, turns into a linear effect in the model of anomalous flow. As said, the linear operator of the anomaly model can be unstable; that is, it can have eigenvalues with positive real part. This is the case for the 10D EOF model that is used here.

We rewrite the model equations (2) in terms of the eigenmodes of $a_{eq}$, yielding a system of five growing/decaying oscillations that are only nonlinearly coupled. Substituting $a = a_{eq} + Sx$ in (2) results in

$$\dot{x} = Bx + \dot{N}(x, x).$$

The (nonorthogonal) matrix $S$ was chosen such that the new linear operator $B$ is block-diagonal: its only non-zero elements are five blocks

$$\begin{pmatrix}
\rho_i & -\omega_i \\
\omega_i & \rho_i
\end{pmatrix}
$$
on the diagonal, with $\rho_i$ and $\omega_i$ the real and imaginary parts of the $i$th eigenvalue pair. The variables $x$ and the parameters of the equation are all real; the $i$th eigenmode is spanned by $x_{2i-1}$ and $x_{2i}$. Turbulent energy in terms of $x$ reads $E = (a_{eq} + Sx)^T (a_{eq} + Sx) = x^T x + a_{eq}^T a_{eq}$.

Numerical integration of the model reveals that the linear terms $Bx$ dominate the behavior of the model. The full model $\dot{x} = Bx + \dot{N}(x, x)$ can be seen as a simple, linear system $\dot{x} = Bx$ perturbed by relatively small nonlinear terms $\dot{N}(x, x)$. Averaged in time, the quantity $|\dot{N}(x, x)|/|Bx|$ is about 1/4. The unperturbed (linear) system already explains much about the intrinsic frequencies of the full (i.e., perturbed) system. However, it does not explain the spectral peak at a frequency of about 0.15 cycles per year. This peak is apparently due to the nonlinear nature of the full model. We therefore need to consider the full nonlinear model in order to understand the ultra-low frequencies that are generated.

A clue as to what is determining the global dynamics of the full model is provided by Fig. 5, in which the squared norm $|x|^2$ is plotted against time during an integration of the full 10D model. This quantity is comparable, but not equal, to the turbulent energy. It can be seen that the system approaches the point $x = 0$ (i.e., the equilibrium $a_{eq}$ when using $a$ as a variable) very closely, time and again. Such behavior suggests the existence of a homoclinic orbit: a phase space trajectory that connects an (unstable) equilibrium point with itself. The presence of such an orbit would provide a good explanation for the departure from and return to (the vicinity of) the equilibrium point by the system, as visible in the 10D model, but also, less pronounced, in the 30D EOF model and in the GCM.

**Fig. 3.** Power spectra of EOF1 in GCM, 30D, and 10D models.
FIG. 4. Eigenmodes of the 10D EOF model linearized around the equilibrium state. Shown are eigenmode patterns (200 mb) (left) at the start and (right) at a quarter of one complete cycle. Contour intervals are 0.05 in nondimensional units.
We shall investigate the hypothesis that the global, long timescale behavior of the 10D model is governed by the presence of a homoclinic orbit in the following section, after providing some theory on homoclinic orbits. We do not attempt to detect this homoclinic orbit in the 10D model (this is a very hard task in 10 dimensions), but rather we focus on a 4D subsystem of the 10D system, which contains the essence of the long timescale dynamics of the 10D model.

5. A 4D chaos generator

Given a set of ODEs \( \dot{y} = G(y, \mu) \) with \( \mu \) denoting the vector of parameters, a homoclinic orbit is a nontrivial solution \( y_{ho}(t) \) of the ODEs, which asymptotically approaches a fixed point \( y_0 \):

\[
\lim_{t \to \pm \infty} y_{ho}(t) = \lim_{t \to \pm \infty} y_0 = y_0.
\]

With \( y_{ho}(t) \) being nontrivial we mean that \( y_{ho}(t) \neq y_0 \) for some values of \( t \). A heteroclinic orbit (or connection) is a solution \( y_{hc}(t) \), which approaches in forward and backward time two different fixed points \( y_+ \) and \( y_-(\neq y_0) \):

\[
\lim_{t \to \pm \infty} y_{hc}(t) = y_+, \quad \lim_{t \to \pm \infty} y_{hc}(t) = y_-.
\]

A heteroclinic cycle is a set of heteroclinic orbits forming a closed loop. The simplest version consists of two heteroclinic orbits: one running from one fixed point \( y_- \) to another \( y_+ \), the second running from \( y_+ \) back to \( y_- \). Strictly speaking, homoclinic and heteroclinic orbits can also exist between other solutions than fixed points (e.g., periodic solutions, invariant tori, etc.), but we will not treat those possibilities here.

Let us focus on the homoclinic case and assume that \( y_{ho}(t) \) exists for the value \( \mu = \mu_{ho} \) of the parameter(s). Generically (assuming nondegenerate and nonzero eigenvalues), there are three possibilities for the leading eigenvalues of the system linearized around \( y_0 \):

1) Two reals: \( (\lambda_1, \lambda_2) \) with \( \lambda_1 < 0 < \lambda_2 \) (the saddle case)
2) One real, one complex pair: \( (\lambda_1, \lambda_2 = i\omega_2) \) with either \( \lambda_1 < 0 < \lambda_2 \) or \( \lambda_2 < 0 < \lambda_1 \) (the saddle-focus case)
3) Two complex pairs: \( (\lambda_1 \pm i\omega_1, \lambda_2 \pm i\omega_2) \) with \( \lambda_1 < 0 < \lambda_2 \) (the double-focus or bifocal case)

In our 10D model, the eigenvalue spectrum of the fixed point consists exclusively of complex pairs, so the eventual homoclinic orbit connected to this fixed point should be of the bifocal type. For the bifocal case it is known that at the bifurcation point \( \mu_{ho} \), as well as in a parameter neighborhood of \( \mu_{ho} \), a countable infinity of periodic orbits (up to arbitrarily high periods) and an uncountable number of aperiodic orbits exist. All these orbits are of saddle-type (i.e., unstable in some phase-space directions). A horseshoe-mapping can be constructed, which means that a chaotic invariant set consisting of the collection of unstable (aperiodic) orbits is created. In our case, the unstable eigenvalues lie closer to the imaginary axis than the stable eigenvalues and most evidence from dynamical systems suggests that the chaotic invariant set is then attracting. In other words, from the eigenvalue spectrum and the (conjectured) existence of the homoclinic orbit, it can be inferred that a chaotic attractor exists for the model under consideration. More detailed information on bifocal homoclinic orbits can be found in Fowler and Sparrow (1991), Glendinning (1997), and Laing and Glendinning (1997). For a more general treatment of homoclinic and heteroclinic bifurcations, see, for example, Kuznetsov (1995) or Wiggins (1988).
The minimum that is needed for such behavior must be four dimensions: two to produce a growing oscillation, two for a decaying oscillation. Such a system can be derived from the 10D system in a crude way by replacing the four dissipating modes by just one: we can isolate a 4D subsystem from the 10D model, consisting of mode 5 (the only growing mode) and one of modes 1–4, and their linear and nonlinear (self-) interactions. The effect of the dissipating mode has then to be identified with the cumulative effect of modes 1–4 in the 10D model; mode 5 is of course identified with itself. It is not necessary to restrict ourselves to triad interactions (as in spectral models) since each interaction between two EOFs already involves many triad interactions between spectral modes.

The choice of the dissipating mode is nontrivial. From the point of view of dynamical systems theory, selecting the leading eigenmodes, hence modes 3 and 5, would be the most natural choice, since the leading eigenmodes determine the geometry of the (phase space) flow close to the fixed point. However, the system consisting of modes 3 and 5 and their interactions does not yield interesting behavior—numerical integration reveals that it ends up in a nontrivial stable fixed point. The combinations 1–5 and 2–5 do not work either: the former also goes to a nontrivial fixed point, the latter goes to unphysically high turbulent energy (a factor 10^5 too high). The only combination that gives similar results as the 10D model is that of modes 4 and 5. The 4D model of these two modes and their interactions reproduces the cycle of growth–transfer–decay; it returns to the fixed point in the origin of phase space again and again, just like the 10D model. Figure 6 makes clear how the total energy is distributed over the dissipating and the growing modes during a burst, both for the 10D model and the 4D model.

Several reasons can be given why the selection of mode 4 gives good results and the selection of the leading stable eigenmode (mode 3) does not. A physical argument is that modes 4 and 5 are both mainly Northern Hemispheric modes, in contrast, mode 3 is mainly tropical. This makes the interaction between 4 and 5 easier than between 3 and 5. From a more mathematical point of view one can observe that the frequencies of modes 4 and 5 are rather close to a 1:3 resonance (i.e., strong resonance) whereas the frequencies of modes 3 and 5 tend toward 1:6, that is, weak, resonance. Strong resonances can be essential for the dynamics of a system. From the theory of nonlinear resonant interactions it is known that, in general, couplings of modes at (or near) low-order (i.e., strong) resonance are the most efficient interactions (see, e.g., Arnol’d 1983). A precise clarification of the influence of the 1:3 resonance is outside the scope of this paper; given the small real parts of the eigenvalues, such a clarification would require knowledge, currently unavailable, of the unfolding of a double Hopf-bifurcation at 1:3 resonance (see, e.g., LeBlanc 2000; Kuznetsov 1995).

An inspection of the amplitudes of all five modes from an integration of the 10D model (not shown) makes clear that mode 4 plays an important role: in the onset of each burst it is the first of the stable modes to be excited by the unstable mode 5. Apparently, the latter transfers its energy more easily to mode 4 than to modes 1, 2, 3. In the decaying phase of each burst, the decay timescale cannot be clearly associated with the stable eigenvalues: the decay is much faster than one would expect from the real parts of the eigenvalues. During the bursts the system is probably too far away from the fixed point for the linear regime to be valid. Further experimentation confirms that mode 4 is crucial in the maintenance of the energy cycle (growth–transfer–decay): the 8D system of modes 1, 2, 3, 5 does not reproduce the cycle (integration results not shown), the 4D system of modes 4 and 5 does.

The 4D subsystem is given by

\[
\begin{align*}
\dot{x}_7 &= \rho_1 x_7 - \omega_4 x_8 + N_7(x_7, x_8, x_9, x_{10}) \\
\dot{x}_8 &= \rho_4 x_8 + \omega_4 x_7 + N_8(x_7, x_8, x_9, x_{10}) \\
\dot{x}_9 &= \rho_5 x_9 - \omega_5 x_{10} + N_9(x_7, x_8, x_9, x_{10}) \\
\dot{x}_{10} &= \rho_5 x_{10} + \omega_5 x_9 + N_{10}(x_7, x_8, x_9, x_{10}), 
\end{align*}
\] (6)

where \(x_7, x_8\) correspond to mode 4 and \(x_9, x_{10}\) to mode
5. The nonlinearities $N_i$ consist again only of quadratic terms. All parameters of the above equations are numerically fixed, due to the empirical nature of the EOF model. If we want to perform a bifurcation analysis we will have to pick our own bifurcation parameter, since common parameters like the Ekman damping rate or orographic scale height are not available anymore. We make the choice of altering $\omega_s$, which has the effect of altering the frequency resonance relation between the two modes. There is no clear physical effect associated with such a change (because of the empirical nature of the model, any chosen parameter will be difficult to interpret physically); however, a changing frequency ratio is likely to cause changes in the model behavior, thereby providing a better understanding of that behavior. It must be mentioned that a frequency-altering parameter also appears as one of the three unfolding parameters for the strongly resonant double Hopf-bifurcation in LeBlanc (2000).

Phase portraits [projected onto the $(x_7, x_{10})$ plane] of numerical integrations with varying $\omega_s$ are shown in Fig. 7 (leaving out initial transient motion). The upper-left panel shows the behavior at $\omega_s = 0.035 343 8$, the original value coming from the EOF model. In Fig. 8 time series of the squared $L_2$-norm, $|\langle x_7, x_8, x_9, x_{10}\rangle|^2$, are shown at the same parameter values. Both chaotic and rather regular (even completely periodic) behavior is found. To get a more detailed picture of the behavior at various values of $\omega_s$, Poincaré sections were made in the half-plane $(x_9 < 0, x_{10} = 0)$. A plot of the $x_9$ value of the intersection points with varying $\omega_s$ (a so-called limit-point diagram, see Fig. 9) exhibits the existence of both chaotic and periodic windows.

The most interesting region of parameter space is the region of decreasing $\omega_s$, that is, $\omega_s < 0.035 343 8$. The model behavior starts to show more and more pronounced bursts, with long periods in between, during which the systems resides very close to the fixed point in the origin (the squared phase space norm dropping below 0.005, see Fig. 8). The behavior stays chaotic but nevertheless becomes very structured. The recurrence of the system, after each burst, to the fixed point in the origin, and the very long time spent extremely close to this fixed point, provides strong evidence for our hypothesis that a homoclinic orbit underlies the observed model behavior.

By carefully inspecting Fig. 9 it was possible to identify an attracting periodic orbit, at $\omega_s = 0.029 018 8$. This orbit, phase-space projections of which are shown in Fig. 10, has a period of 3394 days; it provides a good approximation to the homoclinic orbit. A 1-parameter continuation of the periodic orbit, using AUTO, in order to find the precise location of the homoclinic bifurcation (the point in parameter space where the period of the periodic solution approaches infinity) failed to be conclusive, probably because of numerical problems due to the complex shape of the periodic solution.

The shape of the periodic orbit of Fig. 10 hints at the existence of a second fixed point. This point indeed exists; it has coordinates $(x_7, x_8, x_9, x_{10}) = (0.413 562, -0.650 201, 1.643 48, 1.097 27)$ when $\omega_s = 0.029 018 8$ and is unstable. Its existence and location raise the question whether it is a heteroclinic cycle instead of a homoclinic orbit that structures the system behavior. There are two reasons to hold on to the hypothesis of homoclinic dynamics. First, the time spent near the fixed point in the origin is many times longer (and the proximity many times closer) than the time spent near the second fixed point. At the point $\omega_s = 0.035 343 8$ in parameter space, the fixed point in the origin clearly still plays an important role for the behavior of the system, whereas there seems to be no trace of the other fixed point in the model behavior. Second, since a heteroclinic cycle involves two connections between fixed points, two parameters must in general be tuned in order to arrive at a heteroclinic cycle, rather than one, as is the case for a homoclinic orbit (a heteroclinic cycle is a codimension 2 phenomenon, a homoclinic orbit codimension 1). It is therefore improbable to have encountered a heteroclinic cycle, since only one parameter was varied. It is possible for a heteroclinic cycle to become a codimension 1 phenomenon in a system that possesses symmetry, but our 4D model does not appear to have any symmetry. Nevertheless, it is possible that our model is close to some symmetry that decreases the codimension of a heteroclinic cycle. Breaking such a symmetry can very well break the heteroclinic cycle and produce a homoclinic orbit instead. This interesting possibility lies, however, beyond the scope of this paper.

In conclusion, the existence of a homoclinic orbit of the bifocal type in the 4D model explains the chaotic behavior and the occurrence of very long timescales in the system. The structure of the orbit also explains the observed cycle of growth, transfer, and decay in the system: it is the cycle of the homoclinic orbit that starts at the origin, moves away, and eventually returns to the equilibrium point. The attractor generated around the homoclinic solution will possess roughly this same structure, so the behavior of the system, as it evolves on the attractor, will be marked by the growth-transfer-decay cycle. This is in agreement with our earlier observations (e.g., Fig. 6).

6. Homoclinicity in complex models

A natural question to ask is whether the ULFV-producing mechanism found for the low-order model (generation of chaos due to a bifocal homoclinic orbit attached to an equilibrium state close to the climate mean state) will play a role in more complex models. Will the 30-dimensional version of the empirical EOF model and the GCM possess such an orbit? Obviously, establishing direct numerical evidence for such a conjecture is hardly possible. Nevertheless, the situation for the 4D
Fig. 7. Phase plots ($x_9$ vs $x_{10}$) of 4D model at different values of bifurcation parameter $v_5$: (a) 0.035 343 8, (b) 0.034 343 8, (c) 0.033 343 8, (d) 0.032 343 8, (e) 0.031 343 8, (f) 0.030 343 8, (g) 0.029 343 8, and (h) 0.029 018 8. Integration time is 20 000 days.
Fig. 8. Time series of squared phase space norm \( (|x_7, x_8, x_9, x_{10}|^2) \) of 4D model at different values of bifurcation parameter \( \alpha_1 \) (see Fig. 7 for numerical values). Integration time is 20 000 days.
model is quite generic. The model describes deviations from an equilibrium state close to the atmospheric mean state. The equilibrium is of saddle type, it has both growing and decaying eigenmodes. This mixed stability does not depend on the complexity of the model. Moreover, an atmospheric system linearized around the equilibrium point will generically have complex pairs of eigenvalues, which will cause an eventual homoclinic orbit connected to this equilibrium to be of the bifocal type. Unlike the other two cases (the saddle case and the saddle-focus case), where certain conditions on the eigenvalue spectrum have to be met in order to create chaotic dynamics, the bifocal case of homoclinic orbits always involves the generation of a chaotic invariant set. If this set is attracting, the system will possess chaotic dynamics and nontrivial behavior on timescales of arbitrary length.

The existence of a homoclinic orbit for a model describing anomalous atmospheric flow remains, obviously, the piec de résistance of this analysis. Proving, even numerically, the existence of a homoclinic orbit is a very hard task even in three- or four-dimensional models; for high-dimensional models such as GCMs it is virtually impossible. But carefully looking at the outcome of such complex models may provide some circumstantial evidence. Time series of the turbulent energy of the GCM in the first 30 EOFs (Fig. 1) clearly show significant changes in the distance of the system to the origin of phase space, that is, to the mean state. The recurrence of the system to the vicinity of this mean state (which is likely to be close to some equilibrium solution) hints at the presence of a homoclinic connection. The simple fact that the system moves away from the fixed point due to growing perturbations on the mean flow, but eventually returns to the vicinity of the fixed point, already suggests homoclinicity—it is the only known dynamical mechanism of moving away and returning back to some equilibrium state (disregarding here, for brevity’s sake, the possibility of a heteroclinic cycle). We therefore expect the creation of chaotic behavior in the atmosphere due to homoclinicity to be a general mechanism, not an artefact of the model used.

However, we do not want to state here that homoclinic orbits are likely to be existent in truly high-dimensional, complex, realistic atmosphere models, which contain considerable small-scale detail. Rather, we hypothesize that it is the large-scale atmospheric circulation that is marked by homoclinic behavior. The smaller-scale processes in the atmosphere are likely to destroy the supposed homoclinic connection, but the large-scale behavior will still show traces of the homoclinic dynamics. This interpretation is supported by the comparison of the 10D and 30D EOF models and the GCM. The 10D model shows most clearly the characteristics of homoclinic dynamics; adding smaller-scale patterns obscures this behavior, as can be seen in the 30D model. The GCM still shows large variations in turbulent energy and recurrence to the vicinity of the time mean state if its data is projected onto only the first 30 EOFs (i.e., the largest spatial scales). This recurrence is less clear if all resolved scales are taken into account. In short: the small spatial scales provide perturbations, “noise,” which affect (disturb) the homoclinic behavior of the large-scale atmospheric circulation.

Such interpretation is supported by the results of two previous studies, in which signs of homoclinic behavior can be identified. James et al. (1994) run their T21L5 model for 100 yr and calculate EOFs of the zonal-mean

Fig. 9. Limit-point diagram showing the result of Poincaré sections made at various values of the bifurcation parameter $\omega_5$. 

\begin{center}
\begin{tabular}{c}
\hline
0.5 \\
0 \\
-0.5 \\
-1 \\
-1.5 \\
-2 \\
-2.5 \\
\hline
\end{tabular}
\end{center}
Then, using an integration with imposed sevenfold symmetry in meridional direction (only wave-numbers 0, 7, and 14 are retained), phase-space projections onto the plane of the first two principal components are made that clearly show that the model returns time and again to the same neighborhood (small positive PC1, PC2 almost zero). Note that this neighborhood, at least in that particular projection, is close to the time-mean state, just as in our results. Moreover, projection onto the first two EOFs implies a selection of large spatial scales. Such a selection, or filtering, is likely to enhance the visibility of the homoclinic behavior.

In an entirely different study, directed at the existence of free modes, Branstator and Opsteegh (1989) find orbits recurring to one point in phase space when integrating the barotropic vorticity equation with forcing and dissipation, at T15 resolution. The point to which the system returns is a numerically approximated solution (i.e., equilibrium) of the free (unforced, nondissipative) barotropic vorticity equation. The system seems to be further influenced by another equilibrium state, but it does not approach this second fixed point as close as it approaches the first fixed point. This is reminiscent of our findings for the 4D model and raises again the question to what extent the supposed homoclinic behavior is related to an eventual heteroclinic cycle between two equilibrium states.

It must be stressed that the above studies have a different focus than the one presented here. One major difference lies in the complexity of the models used. Therefore, our interpretation of certain results of James et al. and Branstator and Opsteegh as signs of homoclinic behavior does not necessarily imply that the same physical mechanism(s) can be associated with all three
cases of supposed homoclinic dynamics. Rather, we want to draw attention to a mathematical structure that seems to be also present in other (and more complex) atmospheric models than the low-order EOF model of Achatz and Branstator. The physical properties associated with this mathematical structure need to be further elucidated.

7. A T21 barotropic model

To further support the hypothesis that homoclinic behavior is a generic aspect of models of large-scale atmospheric flow, a standard barotropic spectral model is taken as an example. This model, truncated at T21, describes the winter circulation of the Northern Hemisphere and possesses a realistic climate and reasonable low-frequency variability. The model has 231 dimensions, due to its restriction to one hemisphere. Detailed information on the model can be found in Selten (1995). Here, we mention only the presence of Ekman damping, hyperviscosity (i.e., scale-selective damping) and orography in the model. The model is forced by a constant vorticity forcing that has been tuned in order to get a realistic climate mean state and realistic low-frequency variability.

The model has been integrated for 10 000 days, using a Runge–Kutta fourth-order routine with a 30-min time step and daily output. The timescale of the Ekman damping is set to 15 days; the hyperviscosity is set such that the damping timescale for wavenumber 21 is 3 days. The time mean was calculated and subtracted from the output, resulting in time series containing deviations of spectral coefficients from their time mean value. Using these data, we followed essentially the same steps as previously for the EOF model. A kinetic energy metric was used to calculate the turbulent energy of the model run (i.e., the phase space distance to the climate mean state was calculated using an energy norm). The time series of the turbulent energy is shown in Fig. 11. As before, large variations in turbulent energy can be seen to occur. The model returns, time and again, to the vicinity of the climate mean state.

Furthermore, phase plots were made from one segment of the data: between day 1550 and day 2000, clear transitions in turbulent energy can be observed (also shown in figure 11). A projection of the phase space orbit of the model onto the plane spanned by the real parts of the $Y_3^1$ and $Y_3^2$ spectral coefficients (picked, somewhat arbitrarily, to focus on large-scale patterns) is shown in Fig. 12. To further clarify what is happening, the segment has been cut in three: days 1550–1680 (high turbulent energy), days 1680–1850 (low turbulent energy), and days 1850–2000 (again high turbulent kinetic energy). The projection of these three separate segments are also shown in Fig. 12. The plots clearly show a “trapping” of the system in a rather small region close to the origin (the climate mean state) during days 1680–1850, indicating the presence of a fixed point near which orbits remain for some time. Also, the other two segments (days 1550–1680 and 1850–2000) show large excursions in one specific direction of phase space. Apparently, the excursions are not completely arbitrary but somehow structured; a homoclinic orbit could explain such structure.

The overall picture suggests the influence of a homoclinic orbit in this T21 model of barotropic flow, supporting our hypothesis that the presence of homoclinic dynamics in models of large-scale atmospheric flow is a robust feature, not a model artefact. We expect to report in more detail about the dynamics of the T21 model in the near future.

8. Conclusions

We have studied the dynamics of an efficient 10-dimensional EOF model describing global atmospheric behavior, and found the dynamics on long timescales to be essentially driven by the presence of a homoclinic orbit connecting the equilibrium state close to the climate mean state with itself. The orbit lies at the core
of the growth–transfer–decay cycle observed in the model. This cycle is overpronounced in the model used (due to its simplicity) but is likely to be also present in more complex models of atmospheric flow. Circumstantial evidence for this conjecture can be found in some previous studies, as well as in the observed variances in atmospheric turbulent energy in the 30-dimensional version of the EOF model and in the GCM the EOF model is based on. Further support for these findings comes from an integration of a T21 standard (forced, dissipative) barotropic model for Northern Hemisphere atmospheric flow. Both phase space projections and time series of turbulent energy from this model show signs of homoclinic behavior.

The existence of a homoclinic orbit of the described type (bifocal) always entrains the creation of chaotic behavior, and could thereby explain the ultralow-frequency variability (ULFV) present in "standalone" atmospheric models (without coupling to oceans or other slow climate components).

The analysis and results obtained in this study raise some interesting issues that warrant further study. More evidence for the presence of homoclinic orbits in complex models would support the relevance of our results. The typical cycle of growth–transfer–decay, associated to the homoclinic orbit, has a period of about 6.7 yr in our model. The existence and the period of such a cycle in more complex models are also interesting to consider. The period of 6.7 yr is comparable to periods of ultralow-frequency maxima found in previous studies, but it is unclear whether the atmospheric patterns associated with these peaks are of the same type as the patterns in the 10D EOF model.

Other issues arise out of the observed mathematical structures in the model: the role of the 1:3 resonance in the formation of the homoclinic connection is not resolved yet, nor is the possible link with the existence of a heteroclinic cycle. Such cycles have been observed by Aubry et al. (1988) in the context of efficient models of a fluid-dynamical problem [see also Holmes et al. (1997) and references therein]. Heteroclinicity is generally associated with symmetries in the model; one can wonder whether atmosphere models like the one that was the subject of this paper are close to such a symmetry, and whether it could be connected with the existence of heteroclinic cycle(s). Finally, since not many examples of bifocal homoclinic bifurcations are known in applications, more numerical details of our example would be of interest.

Continuation of the present work can be envisioned...
in two directions. Firmer and more rigorous mathematical support for the conjecture of the influence of homoclinic dynamics in atmosphere models is one path to be taken. The other one should head for a clarification of the physical phenomena that are connected to the homoclinic behavior, and for an understanding of the effect of perturbations (caused by, e.g., small-scale atmospheric processes) on this behavior. We intend to work in both ways.

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